BEURLING-MALLIAVIN THEORY FOR SUB-EXPONENTIAL AND SUPER-EXPONENTIAL GROWTH

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Let $\rho > 0$ and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\gamma'(x) \gtrsim -|x|^{\rho-1}, \qquad (x \to \pm\infty).$$

Consider the Toeplitz operator with symbol $u = e^{i\gamma}$,

$$T[u] : H^2(\mathbb{R}) \to H^2(\mathbb{R}), \qquad f \mapsto P_+(uf).$$

Here $H^2(\mathbb{R})$ is the Hardy space and P_+: $L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is the orthogonal projection. We establish a criterion for the injectivity of $T[u]$, up to multiplication of the symbol by an arbitrarily small power of the "gap" factor

$$M(x) = e^{ix|x|^{\rho-1}}, \qquad (x \in \mathbb{R}).$$

We write $\gamma \in (*)$ if $\gamma(\mp\infty) = \pm\infty$. If $\gamma \in (*)$, then $\mathcal{BM}(\gamma)$ denotes the collection of all complementary intervals of the closed set

$$\{x \in \mathbb{R} : \ \forall t \geq x, \ \gamma(t) \leq \gamma(x)\}.$$

In the statements below, the sums are taken over intervals in $\mathcal{BM}(\gamma)$, l denotes the length of the interval, and d its distance from the origin plus l.

Theorem (sub-exponential case $\rho \leq 1$)**.**

(i) If $\gamma \notin (*)$, *or if* $\gamma \in (*)$ *but* $\sum d^{-2} l^2 = \infty$, *then* $\ker \ T[uM^\epsilon] = 0$ *for all* $\epsilon > 0$.

(ii) If $\gamma \in (*)$ *and* $\sum d^{-2} l^2 < \infty$, *then* $\ker T[uM^\epsilon] \neq 0$ *for all* $\epsilon < 0$.

Theorem (super-exponential case $\rho \geq 1$)**.**

(i) If $\gamma \notin (*)$, *or if* $\gamma \in (*)$ *but* $\sum d^{\rho-3} l^2 = \infty$, *then* $\ker \ T[uM^\epsilon] = 0$ *for all* $\epsilon > 0$.

(ii) If $\gamma \in (*)$ *and* $\sum d^{\rho-3} l^2 < \infty$, *then* $\ker T[uM^\epsilon] \neq 0$ *for all* $\epsilon < 0$.

In the exponential case $\rho = 1$, the choice $U = B\bar{S}$, where $S(x) = e^{ix}$ and B is a Blaschke product, gives the classical BM theorem about completeness of exponentials in $L^2(a, b)$. Our results have the same realm of applications as the BM theory (completeness and minimality of families of special functions, weighted approximation, distribution of zeros of entire functions, gap and density theorems, direct and inverse spectral problems), but these applications concern the theory of Fourier transforms associated with general self-adjoint problems (perhaps with singular endpoints) rather than just the theory of classical Fourier transform.

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